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Effective and Noneffective Results on Certain Arithmetical Functions

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Some Ω results are proved in this paper. It suffices to state a particular result. Write $D(n)$ for the number of square free divisors of n. Next put

$$\sum_{n=1}^{N} D(n) - \frac{x}{\zeta(2)} \times \left(\log x + \left(2\gamma - 1 - \frac{2\zeta'(2)}{\zeta(2)}\right)\right) = E(x),$$

where N is the integral part of x. Given any $\epsilon > 0$ we define explicitly a sequence of intervals in each of which $\max |E(x)| x^{-1/4+\epsilon} > 1$.

1. INTRODUCTION

Let $\{a_n\}$ ($n = 1, 2, 3,\ldots$) be an infinite sequence of complex numbers with $a_n = O_\epsilon(n^\epsilon)$ for every fixed $\epsilon > 0$. Suppose that the Dirichlet series $F(s) = \sum_{n=1}^{\infty} a_n n^{-s}$ ($s = \sigma + it$, $\sigma > 1$) admits of an analytic continuation in $\sigma > 0$ (we can also deal with a less restrictive inequality than $\sigma > 0$) from $\sigma > 1$ along lines parallel to the real axis except on those lines which contain singularities. We assume that the singularities of $F(s)$ are finitely many in every compact subset of $\sigma > 0$. Let C be a simple closed curve (on which $F(s)$ can be continued) with an anticlockwise orientation, an initial point and a final point which is the same as the initial point. We suppose that C is contained in a compact subset of $\sigma > 0$. We now define for all $x > 0$, three functions

$$M(x) = \frac{1}{2\pi i} \int_C F(s) \frac{x^s}{s}\, ds,$$

$$N(x) = \sum_{n \leqslant x} a_n,$$

and

$$E(x) = N(x) - M(x).$$

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Our object in this note is to study the function $E(x)$ for a class of functions $F(s)$. As a first theorem we prove

THEOREM 1. *Let $a \geqslant 2$ be an integer constant, l and m any two complex constants. Let $F(s) = (\zeta(s))^a(\zeta(2s))^l(\zeta(3s))^m$ and let $E(x)$ be defined as before. Then with $\alpha = \frac{1}{2} - 1/2a$, there holds for every fixed $\epsilon > 0$ and for all $X \geqslant X_0 = X_0(a, l, m, \epsilon)$,*

$$\int_X^{X^{(a+1+\epsilon)/2}} | E(x)\, x^{-\alpha} |^2 \frac{dx}{x} > X^{-\epsilon},$$

where X_0 is a positive constant which is effective.

Remark 1. This result may be compared with the ineffective result due to Saffari [6]. He considers the special case $a = 2$, $l = 0$, and $m = -1$ and proves that

$$\limsup_{x\to\infty} \frac{E(x)}{x^\theta} > 0$$

for $\theta = 2/9$ and improves this result by replacing $2/9$ by an ineffective constant which is slightly bigger. Also the remarks that it is very difficult to handle the case $a = 2$, $l = -1$, $m = 0$ and he has no result in this case. Our result gives the optimal value even in the most general case stated in the theorem.

Remark 2. We have avoided too much generalization. We content ourselves by stating some generalities. We can first of all restrict all the hypothesis to $\sigma > \alpha$. Also let $F_1(s),\ldots, F_k(s)$ be Dirichlet series which have constant term 1 and suppose that the coefficients of $F_1(s),\ldots, F_k(s)$, $(F_1(s))^{-1},\ldots,$ $(F_k(s))^{-1}$ be all $O_\epsilon(n^\epsilon)$ for every fixed $\epsilon > 0$. Next let the mean values $(1/T)\int_T^{2T} | F_j(\frac{1}{2} + it)|^2\, dt$ ($j = 1, 2,\ldots, k$) be all $O_\epsilon(T^\epsilon)$ for every fixed $\epsilon > 0$. Let $\varphi(s)$ be a Dirichlet series which has the property that for fixed σ $(0 < \sigma < \frac{1}{2})$, $\int_T^{2T} | \varphi(s)|\, dt \gg T^{3/2-\sigma}$ and $\int_T^{2T} | \varphi(s)|^2\, dt \ll T^{2-2\sigma}$. Then we can take $F(s) = (\varphi(s))^a \prod_{j=1}^k (F_j(sb_j))^{l_j}$, where b_j are integer constants $\geqslant 2$ and l_j are complex constants. The assertion of Theorem 1 holds without any essential modification. We may also consider generalizations of the type $F(s) = \prod_{k=1}^\infty \zeta(ks)$. In this case we can prove for instance that $E(x) = \Omega(x^{1/6-\epsilon})$ for every fixed $\epsilon > 0$, and more precisely an effective result which is analogous to the assertion of Theorem 1. For some amplifications of this remark see Section 3.

Remark 3. It is not very hard to prove assuming the Riemann hypothesis that the integral in Theorem 1 is $O_\epsilon(X^\epsilon)$ for every $\epsilon > 0$ provided C is chosen suitably. For this purpose if $a = 2$, $l = 0$, $m = 0$ we do not need the Riemann hypothesis in the proof of the result just stated. For the proof of these things one starts with the well-known truncated form of Perron's formula, moves the line of integration to $\sigma = \alpha + \epsilon$, and estimates the mean square. We may also prove without assuming any hypothesis that for all X exceeding an effective positive constant

$$(\log X)^{100} > \int_X^{X^{100}} \left| \frac{E(u)}{u^{1/6}} \right|^2 \frac{du}{d} > (\log X)^{-100},$$

where $E(u) = A(u) - \sum_{j=1}^{7} R_j(u)$, $A(u)$ being the number of Abelian groups of order not exceeding u and $R_j(u)$ is the residue of $(u^s/s) \prod_{k=1}^{\infty} \zeta(ks)$ at its pole $s = 1/j$. (We have not tried to economize on the constants in place of 100.) We can also prove similar results on the error terms in the divisor problem and the circle lattice point problem and so on.

Remark 4. It is easy to see starting from $\int_1^\infty (dN(x)/x^s)\, dx$ and integrating by parts that

$$F(s) - \frac{1}{2\pi i} \int_C \frac{F(W)}{s - w}\, dw = s \int_1^\infty \frac{E(x)}{x^{s+1}}\, dx + \text{an entire function}$$

(on the left-hand side we start with s having large real part and continue analytically). It is plain that if the left-hand side has a singularity with real part $> \alpha_0$ then $E(x) = \Omega(x^{\alpha_0})$. Also if a_n are all real and C is symmetric with respect to the real axis then under the conditions just stated we can prove by using Landau's theorem on the singularity of Dirichlet integrals with positive coefficients that even the stronger result $E(x) = \Omega_{\pm}(x^{\alpha_0})$ holds. For such results see a paper by Vaidya [7].

Remark 5. If we examine the proof (of Theorem 1) below we see that the hypothesis $E(x) = O(x^{\alpha-\epsilon})$ in the most general case of the Introduction leads to

$$\frac{1}{T} \int_T^{2T} | F(\sigma_0 + it)|^2\, dt = O(T^{((4-\epsilon)(1-\sigma_0)/2(1-\alpha))^{-1}}), \qquad (\text{for} \quad \alpha < \sigma_0 < \tfrac{1}{2})$$

and so the equestion arises as to lower bounds for the mean square on the left, which contradict the O-estimate. This leads to $E(x) = \Omega(x^{\alpha-\epsilon})$. We consider some related questions in Section 3.

Before leaving the Introduction we propose some research problems which we are unable to solve. These have relavence to Remark 4. For simplicity we take $F(s) = \zeta(s)(\zeta(2s))^{-1}$.

Problems

We define three functions $E_j(X)$ $(j = 1, 2, 3)$ as follows. Let ϵ be a fixed constant $0 < \epsilon < 10^{-8}$. Let $E(x) = \sum_{n \leqslant x} |\mu(n)| - 6x/\pi^2$,

$$E_1(X) = \max_{X \leqslant x \leqslant X^{100}} \left(\frac{|E(x)|}{x^{1/4-\epsilon}} \right),$$

$$E_2(X) = \max_{X \leqslant x \leqslant X^{100}} \left(\frac{E(x)}{x^{1/4-\epsilon}} \right),$$

and

$$E_3(X) = \min_{X \leqslant x \leqslant X^{100}} \left(\frac{E(x)}{x^{1/4-\epsilon}} \right).$$

Find an effective constant $X_0 = X_0(\epsilon)$ such that for all $X \geqslant X_0$, we have

$$E_1(X) > 1, \qquad E_2(X) > 1, \qquad \text{and} \qquad E_3(X) < -1.$$

2. Proof of Theorem 1

We write

$$F(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s} = \int_1^{\infty} \frac{dN(u)}{u^s} = \int_1^{\infty} \frac{dE(u)}{u^s} + \int_1^{\infty} \frac{M'(u)\, du}{u^s},$$

where real part of s exceeds 1 to start with. The last integral is

$$\frac{1}{2\pi i} \int_C \frac{F(W)}{s - W}\, dw = F_0(s), \qquad \text{say;}$$

$F_0(s)$ is regular for all s with real part greater than 1. It can be continued analytically for all s lying outside C and is there $O(|s|^{-1})$ as $|s| \to \infty$. We put $G(s) = F(s) - F_0(s)$ and observe that

$$G(s) = \int_1^{\infty} \frac{dE(u)}{u^s} = F(s) - F_0(s), \qquad F_0(s) = O\left(\frac{1}{|s|}\right) \qquad \text{as} \quad |s| \to \infty.$$

Let $s_0 = \sigma_0 + it$ (σ_0 being a constant satisfying $\alpha < \sigma_0 < \frac{1}{2}$) be any complex number. We introduce three parameters T, Y, and Z such that $Y \geqslant Z \geqslant T \geqslant 100 (\log T)^{20}$. We next write

$$\begin{aligned} G_1(s_0) &= \frac{1}{2\pi i} \int_{\operatorname{Re} W=2} G(s_0 + W)\, Y^W \Gamma(W)\, dW \\ &= \int^{\infty} \frac{e^{-u/Y}}{u^{s_0}}\, dE(u) \end{aligned}$$

$$= \int_1^Z \cdots + \int_Z^{Y(\log Y)^2} \cdots + \int_{Y(\log Y)^2}^{\infty} \cdots$$
$$= G_2(s_0) + G_3(s_0) + G_4(s_0), \qquad \text{say.}$$

It is easily verified that

$$\int_T^{2T} |\, G_2(s_0)|^2 \, dt = O(Z^{2-2\sigma_0}(\log Y)^\lambda) \qquad (\text{where } \lambda > 1 \text{ is a constant}),$$

and

$$\int_T^{2T} |\, G_4(s_0)|^2 \, dt = O(1).$$

Let us note that we can assume $\int_U^{2U} (|\, E(u)|^2 \, du/u^{2\alpha+1}) < 1$ for every U satisfying $\frac{1}{2}Z \leqslant U \leqslant 2Y(\log Y)^2$. Otherwise we will see that (by choosing Z and Y suitably) there is nothing to prove. Hence we can select Z and Y without affecting their order of magnitude so that $E(u) = O(u^\alpha)$ holds when $u = Z$ and $u = Y(\log Y)^2$. Thus we see that

$$G_3(s_0) = O(1) - \int_Z^{Y(\log Y)^2} E(u) \frac{d}{du} \left(\frac{1}{u^{s_0}} e^{-u/Y} \right) du.$$

Modifying the standard techniques for mean value of Dirichlet polynomials to Dirichlet integrals it is a simple exercise (see, for instance, Lemma 2 at the end of Section 2) to prove

$$\int_T^{2T} |\, G_3(s_0)|^2 \, dt = O\left(T^2(\log Y)^{20} \int_Z^{Y(\log Y)^2} \frac{|\, E(u)|^2}{u^{2\sigma_0+1}} \, du \right).$$

Assuming now that $\int_Z^{Y(\log Y)^2} (|\, E(u)|^2 \, du/u^{2\alpha+1}) < T^{-\epsilon}$, we have

$$\int_T^{2T} |\, G_1(s_0)|^2 \, dt = O\left(\left(Z^{2-2\sigma_0} + \frac{T^{2-\epsilon}}{Z^{2(\sigma_0-\alpha)}} \right) (\log Y)^{20\lambda} \right).$$

We select Z by $Z^{2-2\alpha} = T^{2-\epsilon}$, i.e., $Z = T^{(2-\epsilon)/(2-2\alpha)}$ and see that the O-term is

$$O(T^{(2-\epsilon)(1-\sigma_0)/(1-\alpha)}(\log Y)^{20\lambda}).$$

On the other hand we prove that $\int_T^{2T} |\, G_1(s_0)|^2 \, dt \geqslant T^{1+2a(1/2-\sigma_0)-\epsilon^{10}} = T^{a+1-2a\sigma_0-\epsilon^{10}}$ for a choice of Y not bigger than $20T^{a+\epsilon}$ and $T \geqslant T_0$ which is an effective constant. If we agree to write $A \doteq B$ for the statement $(1/40)A \leqslant B \leqslant 40A$ then our choice of Y and Z will be given by $Z \doteq T^{(2-\epsilon)/(2-2\alpha)}$, $Y \doteq T^{a+\epsilon}$ and $Y \doteq Z^{(2-2\alpha)(a+\epsilon)/(2-\epsilon)}$. It suffices to choose σ_0 such that

$$\frac{(2-\epsilon/2)(1-\sigma_0)}{1-\alpha} = a + 1 - 2a\sigma_0 \, .$$

This leads to a $\sigma_0 > \alpha$ and $<\frac{1}{2}$. This would prove that

$$\int_Z^{Y(\log Y)^2} \frac{|E(u)|^2}{u^{2\alpha+1}} du > T^{-\epsilon}$$

for all $T \geqslant T_0$, an effective constant. This would complete the proof of Theorem 1. We have still to prove the following

LEMMA 1. *For $T \geqslant T_0$ which is an effective constant,*

$$\int_T^{2T} |G_1(s_0)|^2 dt \geqslant T^{1+2a(1/2-\sigma_0)-\epsilon^{10}}.$$

Proof. Let $\sigma_1 = \frac{1}{2}(\sigma_0 + \alpha)$. Divide the line $\sigma = \sigma_1$, $T \leqslant t \leqslant 2T$, into intervals J (which are nonoverlapping and cover the maximum possible length) of length $(\log T)^{20}$. Let $M(J)$ denote the maximum of $|\zeta(s)|$ for s in J. We know that $\sum_J (M(J))^2 = O(T^{2-2\sigma_1}(\log T)^{40})$. We omit those intervals J for which $M(J) > T^{1/2-\sigma_1}(\log T)^{100}$. There are at most $O(T(\log T)^{-160})$ of these omitted intervals. Some of the rectangles defined by t in J and $\frac{1}{2}(\sigma_1 + \alpha) \leqslant \sigma \leqslant 1$ may contain zeros of $\zeta(2s)\,\zeta(3s)$. The number of rectangles which may contain a zero of $\zeta(2s)\,\zeta(3s)$ is $O(T(\log T)^{-160})$ by the standard density theorems for the zeros of $\zeta(s)$. From the remaining intervals J we remove bits of length $(\log T)^3$ on either side (i.e., both above and below) and denote the resulting intervals by I. The total length of intervals not covered by I is $O(T(\log T)^{-17})$. On the intervals I it is easy to see that $F(s) = O(T^{a(1/2-\sigma_1)+\epsilon^{20}})$. We go back to the expression for $G_1(s_0)$ with $s_0 = \sigma_0 + it$ and t in I. We break off the portion $\operatorname{Im} W \geqslant (\log T)^2$ of the integral for $G_1(s_0)$ and move the remaining portion of the integral to $\operatorname{Re}(s_0 + W) = \sigma_1$. We see that if we select Y to lie between two constant multiples of $T^{a+\epsilon}$ (say $Y \doteq T^{a+\epsilon}$) then

$$G_1(s_0) = F(s_0) + O(T^{a(1/2-\sigma_1)+\epsilon^{15}-(a+\epsilon)(\sigma_0-\sigma_1)})$$

and so

$$2\sum_I \int_I |G_1(s_0)|^2 dt > \sum_I \int_I |F(s_0)|^2 dt + O(T^{1+2a(1/2-\sigma_1)+\epsilon^{14}-2(a+\epsilon)(\sigma_0-\sigma_1)}).$$

Now $F(s_0)^2 \gg T^{-\epsilon^{13}} |\zeta(s_0)|^{2a}$ and

$$\sum_I \int_I |\zeta(s_0)|^2 dt = \int_T^{2T} |\zeta(s_0)|^2 dt + O\left(\left(\frac{T}{(\log T)^{17}}\right)^{1/2} (T^{3-4\sigma_1})^{1/2}\right)$$

$$\gg T^{2-2\sigma_0}$$

and also by Hölder's inequality

$$\sum_I \int_I |\zeta(s_0)|^{2a}\, dt \gg T^{1+2a(1/2-\sigma_0)}$$

where the constants implied by $\gg$ are all effective. Thus we see that for $T \geqslant T_0$, which is effective

$$\int_T^{2T} |G_1(s_0)|^2\, dt > T^{1+2a(1/2-\sigma_0)-\epsilon^{10}}.$$

This proves Lemma 1.

We next prove the following lemma only for the sake of completeness.

LEMMA 2. *If $B(u)$ is any integrable complex-valued function of u $(1 \leqslant u \leqslant x)$ and $\psi(t) = \int_1^x B(u)\, u^{it}\, du$ then for $T \geqslant 3$,*

$$\int_0^T |\psi(t)|^2\, dt \leqslant (T + O((x+3)\log(x+3))) \int_1^{x+3} |B(u)|^2\, du,$$

where the O-constant is absolute.

Remark. It is possible to replace the R.H.S by $\int_1^{x+3} (T + O(u))\, |B(u)|^2\, du$.

Proof. We can assume that x is an integer by defining $B(u)$ to be zero in the necessary range. We can now write

$$\psi(t) = \sum_{n=1}^{x-1} \int_n^{n+1} B(u)\, u^{it}\, du = \int_0^1 \left(\sum_{n=1}^{x-1} B(n+u)(n+u)^{it} \right) du.$$

By Hölder's inequality,

$$\int_0^T |\psi(t)|^2\, dt \leqslant \int_0^1 \left(\int_0^T \left| \sum_{n=1}^{x-1} B(n+u)(n+u)^{it} \right|^2 dt \right) du.$$

By using

$$\left| \log \frac{m+u}{n+u} \right| \gg \frac{|m-n|}{m+n}$$

and some easy computations we are led to the expression

$$(T + O((x+3)\log(x+3))) \sum_{n=1}^{x-1} |B(n+u)|^2$$

for the inner integral. This proves the lemma.

3. Some Amplifications of Remarks 2 and 5

We begin by saying that for some class of examples of functions like $\varphi(s)$ one may consult [2–4]. We next divide this section into some subsections.

(i) To amplify our remark 2 we note that $\int_T^{2T} |\varphi(s)|^2\, dt$ does not exceed a constant multiple of $T^{2-2\sigma}$ the constant depending continuously on σ (for σ in any closed interval not containing $\frac{1}{2}$). This implies that if we divide $(T, 2T)$ into intervals I of equal length ($\geqslant 8$) then on most of the subintervals $\int_I |\varphi(s)|^2\, dt = O(|I|\, T^{1-2\sigma})$ uniformly in σ (σ being confined to any interval, $|I|$ denotes the length of I). From this it follows that if I' denotes the interval I with intervals of length 1 removed on either side then for most of the subintervals I', $\max_{s\in I'} |\varphi(s)| = O(|I|^{1/2}(\log T)^2\, T^{1/2-\sigma})$. We state the remainder of the details in a qualitative way. We are led to the following question: Given a set of disjoint intervals I_1 of equal length contained in $(T, 2T)$ we suppose that the intervals I_1 together cover the whole of $(T, 2T)$ except bits of total length $(T(\log T)^{-17})$. We ask what functions $\varphi_1(s)$ satisfy

$$\frac{1}{T}\sum_{I_1}\int_{I_1} |\varphi_1(s)|\, dt > T^{a(1/2-\sigma)-\epsilon}$$

for at least one σ in $\sigma_3 \leqslant \sigma \leqslant \sigma_4$ where $\sigma_4 < \frac{1}{2} - 2\epsilon$, and $\sigma_4 - \sigma_3 < \epsilon^{20}$? This would suffice to prove the theorem with the modifications. It is trivial to check that $\varphi_1(s) = (\varphi(s))^a$ satisfies the requirement. We can also take $\varphi_1(s) = (\varphi(s))^{a_1}$ times a power product (with nonnegative integral exponents) of certain functions which we describe below. Here $a = a_1 +$ the sum of the other exponents. The functions should be selected from zeta function, L-series, Hurwitz zeta function (i.e., $\sum_{n=1}^{\infty} (bn + d)^{-s}$, where b and d are rational and positive), Abelian L-series associated with a quadratic field, zeta function of a ray class in a quadratic field. Also to each factor of which $\varphi_1(s)$ is a power product we may add any finite Dirichlet series of boundedly many terms and modify $\varphi_1(s)$ accordingly. Next in place of $(\varphi(s))^{a_1}$ we may also take a power product $(\varphi^{(1)}(s))^{a_{11}}(\varphi^{(2)}(s))^{a_{12}}$ where a_{11} and a_{12} are nonnegative integers with sum a_1 and $\int_T^{2T} |\varphi^{(1)}(s)\, \varphi^{(2)}(s)|\, dt$ is $\gg T^{2-2\sigma}$ (for $\sigma < \frac{1}{2}$) and $\int_T^{2T} |\varphi^{(j)}(s)|^4\, dt \ll T^{3-4\sigma}$ (for $\sigma < \frac{1}{2}$). Finally some more generalizations are possible and we do not wish to state them here.

(ii) Theorem 1 has its more general analogs if terms like $(\zeta(2s))^l$ $(\zeta(3s))^m$ are absent. For instance, we can take $F(s)$ to be a finite power product (with nonnegative integral exponents) of Dirichlet series each of which has a functional equation from s to $1 - s$. All that we want is the regularity of those Dirichlet series in $t \geqslant 20$, $\sigma \geqslant -2$, and the estimate $|F(s)|$ is $\gg$ and $\ll t^{a(1/2-\sigma)}$ in $-2 \leqslant \sigma \leqslant -1$.

(iii) If we are content with an ineffective result it is much simpler. Because, the contradiction to the Ω result which we wish to prove secures the necessary regularity conditions on $F(s)$ and also an ovbious upper bound for $\int_T^{2T} | F(s)|^2 \, dt$ ($\sigma < \frac{1}{2}$). So we have to concentrate mainly on a lower bound for this integral. In this connection we are led to the following general problem.

(iv) *Problem*: Let $f(s) = 1 + \sum_{n=2}^{\infty} a_n \lambda_n^{-s}$ (where $\{\lambda_n\}$ is a sequence of real numbers such that $a_n = O_\epsilon(n^\epsilon)$, $1 < \lambda_3 < \lambda_4 < \cdots$, $\lambda_{n+1} - \lambda_n \gg 1$ and $\lambda_n = O(n)$) be a generalized Dirichlet series which can be continued analytically in $\sigma \geqslant \frac{1}{5}$, $t \geqslant 20$ and there $f(s) < t^A$. For various sequences $\{a_n\}$, $\{\lambda_n\}$ (A may depend on these) we get various functions whenever possible. We take the class of all such functions $f(s)$. Let $100 \leqslant H \leqslant T$ and $\mu \geqslant 0$ a real constant. We define

$$M_{f,\mu} = (f, \sigma, T, T + H, \mu) = \frac{1}{H} \int_T^{T+H} |f(\sigma + it)|^\mu \, dt.$$

Let πf^μ be a finite power product (with real nonnegative constant exponents) of functions of the class. With it we associate

$$\begin{aligned} M_{(f),(\mu)} &= M((f), \sigma, T - H^{1/2}, T + H + H^{1/2}, (\mu)) \\ &= \frac{1}{H} \int_{T-H^{1/2}}^{T+H+H^{1/2}} | \pi f^\mu(\sigma + it)| \, dt. \end{aligned}$$

The problem is to give the lower bound $M_{(f),(\mu)} \gg \pi(M_{f,1})^\mu$ where the constant is independent of T and H.

As far as we are aware nothing in this direction is known even for the product of two ordinary Dirichlet series.

(v) Some Partial Solutions. *Let* $100 \leqslant (\log T)^{1/100} \leqslant H \leqslant T$ *and define instead of* $M_{f,\mu}$ *the function* $M^*_{f,\mu}$ *as follows. Divide the range* T, $T + H$ *into intervals* J_1 *of length* 1 (*ignoring a bit at one end*). *Instead of the integral define* (*for any fixed* σ)

$$M^*_{f,\mu} = \frac{1}{H} \sum_{J_1} \max_{t \in J_1} | f(s)|^\mu$$

and similarly the function $M^*_{(f),(\mu)}$. *Then it is possible to prove convexity results for these functions which imply*

$$M^*_{(f),(\mu)} \gg H^{-\epsilon}$$

for every fixed $\epsilon > 0$ *and the constants involved are effective.*

Sketch of proof. It is probably well known that the value of $\sum \mu \log | f(s)|$ at any point does not exceed its average (times the radius) over any circle with this point as center (see [5], in particular the proof there of Lemma 2).

This gives maximum modulus principle for the functions of the type $P = \pi f^{\mu}$ for circles and so for disks the maximum modulus cannot be attained in the interior. This follows from the fact that the power product just mentioned is constant if its absolute value is constant over a domain. Next, maximum modulus principle can be upheld even for arbitrary domains granting it for disks.

We now refer to the argument on page 179 in [1] for details. In place of the auxiliary function (24) there we consider

$$P(s)\, e^{(s-s_0)^{4m+2}} Z^{s-s_0}$$

where m is a large integer constant. Applying maximum modulus principle to a suitable rectangle we get the partial solution stated above.

We have some other things to say but we stop here since they are not very interesting. The only thing we may say is that the functions $M_{f,\mu}^{*}$ and so on are related by inequalities both ways to $M_{f,\mu}$ and so on introduced already.

Note added in proof. For the circle and the divisor problems, Cramer [8] has proved more precise Ω results. The most recent result is due to K. S. Gangadharan, Two classical lattice point problems, *Proc. Cambr. Phil. Soc.* **57** (1961), 699–721. The present paper is of interest because of its generality. The present work is continued in two further papers II and III with the title 'Some problems of analytic number theory' by us. In particular we solve the problems mentioned before Section 2 of the present paper in sufficient generality and sharper form.

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